

# Structure of force networks in tapped particulate systems of disks and pentagons.

## I. Clusters and loops

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The force network of a granular assembly, defined by the contact network and the corresponding contact forces, carries valuable information about the state of the packing. Simple analysis of these networks based on the distribution of force strengths is rather insensitive to the changes in preparation protocols or to the types of particles. In this and the companion paper [Kondic *et al.*, *Phys. Rev. E* **93**, 062903 (2016)], we consider two-dimensional simulations of tapped systems built from frictional disks and pentagons, and study the structure of the force networks of granular packings by considering network's topology as force thresholds are varied. We show that the number of clusters and loops observed in the force networks as a function of the force threshold are markedly different for disks and pentagons if the tangential contact forces are considered, whereas they are surprisingly similar for the network defined by the normal forces. In particular, the results indicate that, overall, the force network is more heterogeneous for disks than for pentagons. Such differences in network properties are expected to lead to different macroscale response of the considered systems, despite the fact that averaged measures (such as force probability density function) do not show any obvious differences. Additionally, we show that the states obtained by tapping with different intensities that display similar packing fraction are difficult to distinguish based on simple topological invariants.

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### I. INTRODUCTION

Particulate systems are very common in nature and in a variety of technologically relevant applications. Many of these systems are composed of particles that remain in contact for relatively long periods. These contacts form a network, whose properties are important for the purpose of understanding the system as a whole. However, the contact network provides only partial information about the interaction between the particles. In order to obtain a deeper understanding of a particulate system, the strength of the contacts needs to be considered. This naturally leads to the concept of force networks. The properties of these mesoscale structures are of fundamental importance for the purpose of revealing the underlying physical causes of many phenomena. For example, the electrical conductivity of a granular bed strongly depends not only on the actual strength of the contacts [1], but also on the structure of the contact network and the presence of paths of strong contacts [2]. Similarly, the elastic properties of these systems are very sensitive to the characteristic of underlying force network [3,4], to the degree that different packing structures can be distinguished by their response to sound propagation [5].

Due to their importance, both contact and force networks have been analyzed extensively in recent years. While earlier research focused mostly on basic statistical properties of these networks by computing probability density functions (PDFs) of the contact forces (see, e.g., Refs. [6,7]) during the last few years a variety of other approaches have been considered. These approaches include, among others, network type of analysis [8–11], exploration of the properties of the cycles

(loops) formed by the force [12] and contact [13,14] networks, and the force tiling approach [15]. Regarding the connection (or the lack of it) between force and contact networks, it is worth mentioning recent works [16,17] that illustrated that the properties of “force chains” (defined appropriately) differ significantly from the dominant geometrical features (bridges or arches) arising in the contact network. The analysis of topological properties of force and contact networks [18–21] has quantified the differences between these networks in much more detail and has generally shown that the properties of the force networks depend strongly on the force level (threshold) considered.

Particulate systems consisting of circular or spherical particles have been intensively studied by theoretical, computational, and experimental approaches. The simple shape of these particles makes the study easier compared to the systems consisting of particles of other shapes. However, the particles relevant to applications are typically not circular. Therefore, it is important to understand similarities, as well as differences, between the systems consisting of circular or spherical particles and particles of other shapes. This problem appears to be rather complex, and the number of works considering in detail the influence of the particle shape is rather limited, with a focus on PDFs of the interparticle forces [22,23], contact networks [24], anisotropy of force networks under shear [25–27], and influence of particle shape on shear strength of the material [28]. Connectivity properties were also discussed in the context of experiments carried out with photoelastic particles [29]. Considering the force PDF for a moment, we note that this measure provides only information about the strength of the forces and does not shed much light on

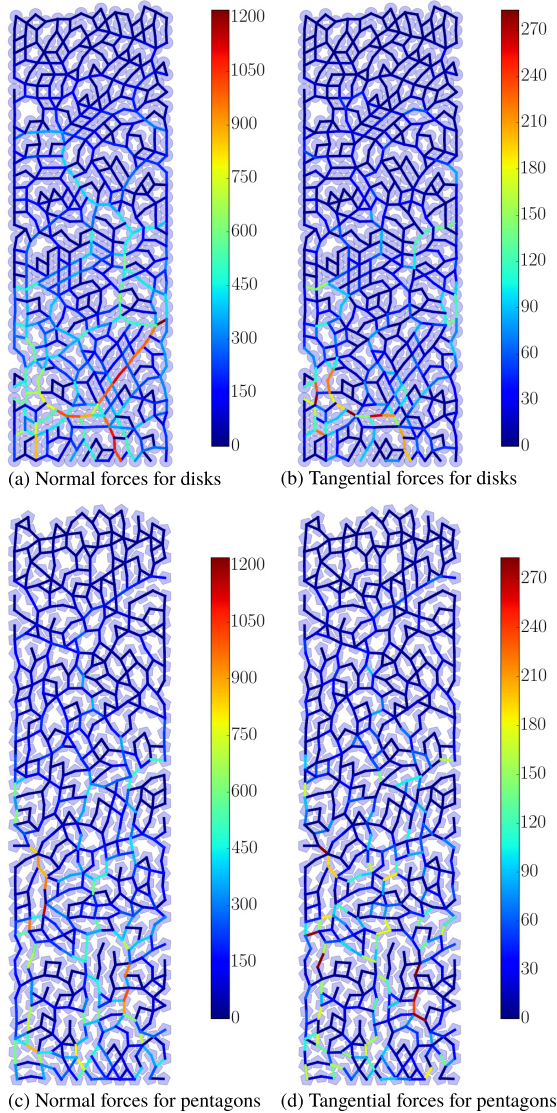


FIG. 1. Sample snapshots of the disks (a) and (b) and pentagons (c) and (d) prepared by tapping at low tapping intensity,  $\Gamma = 3.83\sqrt{dg}$  (see Sec. II). The contact forces are indicated by segments connecting the touching particles colored according to the strength of the normal (a) and (c) or tangential (b) and (d) components of the force. The color scale indicates the force in units of  $mg$ , with  $m$  the mass of a particle and  $g$  the acceleration of gravity.

the local and global properties of the underlying force network. We will show that physical systems with similar PDFs can give rise to very different force networks and therefore could exhibit different response to external perturbation.

In this and the companion paper [30], we focus on quantifying force networks in tapped systems of disks and pentagons in two spatial dimensions. Figure 1 shows examples of the force networks that we will consider. In addition to quantifying the influence of particle shape on the properties of force networks, we also consider the spatial heterogeneity induced by gravity, as well as the effect of the preparation protocol on seemingly equivalent states. For this purpose, we use the techniques based on computational topology that have been already used

to analyze systems exposed to compression [18–21]. In the present paper, we will consider simple measures, based on analyzing the components (clusters) and the loops (holes) formed by the force networks at different force thresholds. We focus on the Betti numbers  $\beta_0$  and  $\beta_1$ , which correspond to the number of components and loops, respectively. Additional more complex measures based on persistence homology are discussed in Ref. [30].

One interesting feature of granular columns subjected to vibration (or taps) is that they reach steady states that may have the same packing fraction,  $\phi$ , even if the intensity of the taps differ significantly [31–33]. This is the case also for polygonal grains [34]. Hence, a natural question is what distinguishes two states with same  $\phi$ . Here we will go beyond some preliminary findings [14] on the distinctive contact network of these same- $\phi$  steady states and look into the associated force network.

After presenting the computational methods in Sec. II, we compare force networks in different settings. In Sec. III A we show that the gravitational compaction influences the structure of the force networks beyond a simple change in the average force level. In Sec. III B we turn our attention to comparing the force networks of a system characterized by the same packing fraction obtained with different tap intensities. Then, in Sec. III C, we identify similarities and differences between force networks corresponding to the systems built out of disks and pentagons. Section IV is devoted to conclusions and future outlook.

## II. METHODS

### A. Simulation

We carried out discrete element method (DEM) simulations of two-dimensional (2D) systems of particles. The simulations were implemented by means of the Box2D library [35]. Box2D uses a constraint solver to handle hard bodies. At each time step of the dynamics a series of iterations (typically 20) are used to resolve overlaps between bodies through a Lagrange multiplier scheme [36]. After resolving overlaps, the inelastic collision at each contact is solved and new linear and angular velocities are assigned. The equations of motion are integrated through a symplectic Euler algorithm. Solid friction is also handled by means of a Lagrange multiplier scheme that implements the Coulomb criterion. This library achieves a high performance when handling complex bodies such as polygons. The approach yields realistic dynamics for granular complex bodies and has been successfully used to study grains under tapping [34,37] and also under vigorous vibration [38].

The systems consist of 500 monosized particles (either disks or regular pentagons). Both type of particles have the same area and material density. The diameter,  $d$ , of the disks is set to 1.0. Pentagons have a radius (center-to-vertex distance) of  $0.57474d$ ; this choice ensures that they have the same mass as disks. The particles are placed in a rectangular box  $14.4d$  wide, which is confined to move in the vertical direction. This box is high enough to avoid particles’ contacts with the ceiling.

We set the particle-particle interactions to yield a low normal restitution coefficient  $\epsilon = 0.058$ . This ensures that the grains come to rest after each tap in a relatively short simulation time. The static,  $\mu_s$ , and dynamic,  $\mu_d$ , friction coefficients

are set to 0.5. The particle-wall restitution coefficient is as in the particle-particle interaction. The particle-wall friction coefficients are  $\mu_s = \mu_d = 0.005$ . These low values lead to a reduced Janssen-like effect. We define  $m$  as the mass of a particle and  $g$  as the acceleration of gravity. The time step  $\delta t$  used to integrate the equations of motion is  $0.015\sqrt{d/g}$ .

Particles, initially placed at random, without overlaps in the box, are let to settle until they come to rest in order to prepare the initial packing. Then 600 taps are applied to each sample. After each tap, we wait for the particles to equilibrate. The new static configuration after each tap is saved for the force network analysis. This includes the particle positions and orientations along with the contact forces. In the case of pentagons, Box2D represents side-to-side contacts as two effective point contacts that define a segment along the shared side. We add these two forces on both point contacts to represent the total force exerted on the face-to-face contact. The initial 100 taps are discarded, and only the final 500 ones are used in the analysis. In previous works [32,33], we found that hundred taps was sufficient to reach a well-defined steady state for the tap intensities used.

Tapping itself is simulated by hitting the containing box upwards so it flies and falls back due to gravity on a solid base. To achieve this, while the box is at rest on the solid base, we reset the velocity,  $v_0$ , of the box to a given positive value and restart the dynamics. The box-base restitution coefficient is set to zero. While the box dissipate all its kinetic energy when contacting the base, particles inside the box bounce against the box walls and floor until settling. After all particles come to rest, a new tap is applied. The intensity,  $\Gamma$ , of the taps is measured by the initial velocity imposed to the confining box at each tap (i.e.,  $\Gamma = v_0$ ).

Figure 2 shows the steady state  $\phi$  as a function of  $\Gamma$ . Although pentagons display much lower packing fractions, both systems reach a minimum in  $\phi$ , as discussed in Ref. [34]. The minimum packing fraction results from the competition between arch formation and breakage. At high tap intensities, particles deposit mostly sequentially since they are separated by large distances during the tap, preventing arch formation (a cooperative phenomenon) and leading to relatively high  $\phi$ . At

low tap intensities, any arch formed can be slowly rearranged and broken after a number of taps; however, it can hardly be rebuilt due to the little free volume created by the small taps, which also leads to high packing fractions. The minimum in  $\phi$  is observed at intermediate tapping intensity, where the system attains a dynamic equilibrium between arch formation and breakage with a maximum number and size of arches (see Ref. [31]). In this paper we consider two values of  $\Gamma$ ,  $\Gamma = 3.83\sqrt{dg}$  (low tapping intensity), and  $\Gamma = 12.14\sqrt{dg}$  (high tapping intensity). The low tapping intensity is low enough to warrant that for both particle shapes the minimum  $\phi$  has not been reached, yet large enough to avoid the ergodicity breaking observed at very low tap intensities [39,40]. Due to ergodicity breaking, at lower tap intensities, two independent realizations of the tapping protocol may lead to different, distinguishable, steady states.

Figure 1 shows few snapshots of disks and pentagons (particles and force networks). Since, based on visual inspection, the geometrical arrangement and the forces changes dramatically from tap to tap, all our results are computed by averaging over a large number (500) of taps. This number of realizations is sufficient to decrease the statistical fluctuations significantly and will allow us to identify the differences between the considered networks that are robust with respect to statistical fluctuations. We consider only the particle-particle contacts in our analysis; the contacts with the walls and floor of the container are disregarded.

## B. Betti numbers

The force network is represented by a scalar function,  $f$ , from the contact network to the real numbers. Values of this function at the edges of the network (i.e., the connecting lines between contacting particles) are given by the magnitude of the normal (tangential) force. The function  $f$  is normalized by the average force level in the system. So an edge in the network with the value 1 represents a contact between two particles with a force value equal to the average force  $\langle f \rangle$ .

We are interested in the structural properties of the part of the force network on which  $f$  exceeds a given threshold  $F$ . We study the properties of those parts of the network for different values of  $F$ . For simplicity, we restrict our attention to the number of connected components (just “components” or “clusters” in what follows) and the number of loops present in the networks. In particular, the function  $\beta_0(F)$  measures the number of components for the part of the force network exceeding the force level  $F$ . For  $F > 1$  these components can be thought of as “force chains.” Additional insight into the properties of a force network can be obtained by considering the number of loops,  $\beta_1(F)$ , inside the part of the network exceeding the force level  $F$ . A loop in the network is a closed path of the edges connecting centers of the particles. If not indicated otherwise, we do not consider the trivial loops (i.e., loops made by three contacting particles); furthermore, all the results for  $\beta_0$  and  $\beta_1$  are normalized by the number of particles in the domain used to define the force network under consideration.

A more detailed description of Betti numbers can be found in Ref. [18], and in-depth discussion of computational homology in the context of particular matter is given in Ref. [21].

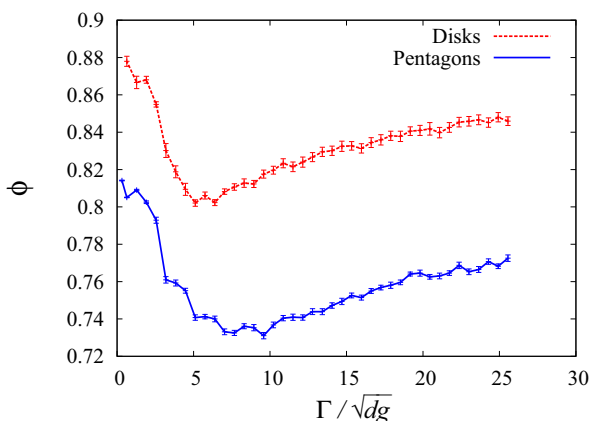


FIG. 2. Steady state packing fraction,  $\phi$ , as a function of  $\Gamma$  for disks and pentagons. The error bars indicate the standard error on the mean  $\phi$ , which is averaged over 30 taps after reaching the steady state.

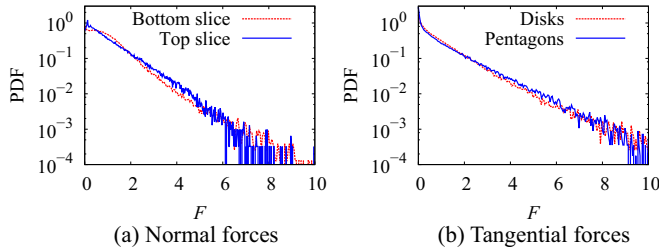


FIG. 3. Probability density functions (PDF) (disks, low tapping).

General treatment of computational algebraic topology can be found in Refs. [41,42].

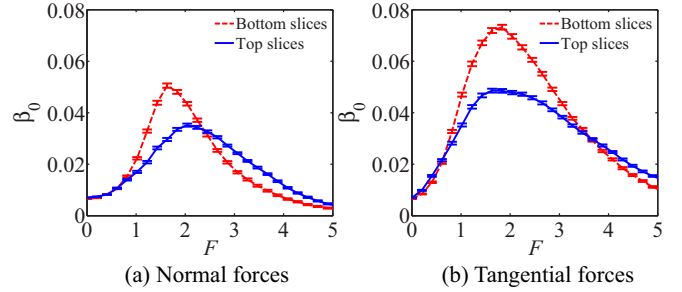
### III. RESULTS

#### A. Heterogeneity of force networks in gravitationally compacted systems

We start by investigating the influence of gravity on the structural properties of the force networks corresponding to the steady states of the tapped disk-based systems. The average contact force,  $\langle f \rangle$ , increases with depth as long as Janssen's saturation is not achieved. As shown in Ref. [43], achieving this saturation is difficult in two dimensions. Therefore, the average force as well as the structure of the force network might change with depth. In order to understand the influence of the depth on the network structure, we compare two distinct horizontal slices of the system. The thickness of both slices is  $10d$ , and the forces in each slice are normalized by  $\langle f \rangle$  inside of the slice. We consider only the particles whose centers are inside the slice and the interactions between these particles. The contacts with other particles or walls are not taken into account. Each slice contains roughly 140 particles, with some variation from tap to tap and from disk-based to pentagon-based systems. The top slice is centered at  $27d$  from the box floor, which ensures that the upper boundary of this slice is about  $4d$  below the free surface. To minimize the boundary effects, the bottom slice is centered at  $7d$  from the bottom of the box. Thus, approximately three bottom layers of grains are excluded.

Figure 3 shows PDFs of normal and tangential forces for the top and the bottom slice of the disk-based system. The comparison of the PDFs for different slices does not reveal any prominent differences on a statistical level. The only noticeable difference can be found at low normal forces where the PDF of the bottom slice shows a plateau. This behavior is expected since gravity leads to compression, and it is known that the PDFs for granular systems exposed to stronger compression show a plateau at small forces; see, e.g., Ref. [20].

Now we turn our attention to structural properties of the force networks for the two slices. We investigate changes in the number of components, present in the force network, as we change the force threshold. Figure 4 shows  $\beta_0$ , as a function of force threshold for the top and bottom slices. For small force threshold,  $F$ , all particles are connected and  $\beta_0 = 1$ . If  $F$  is large, then there are very few contacts experiencing a force larger than  $F$ . This leads to a small number of clusters (small value of  $\beta_0$ ). As shown in Fig. 4, the number of components reaches the maximum for an intermediate value of  $F$ . This value is similar for the top and bottom slices ( $F \approx 2$ ). However,


 FIG. 4.  $\beta_0$  (disks, low tapping). Error bars correspond to the standard error estimated as  $std/\sqrt{N}$  with  $std$  the standard deviation and  $N = 500$ , the number of realizations considered.

the maxima differ significantly. For normal forces there is a much larger number of components in the bottom slice for  $F \in [1, 2]$ . For the tangential forces this is true even for a wider range of  $F$ .

Due to gravitational compaction, the average pressure is larger in the bottom slice. We conclude that this increase in the pressure leads to the formation of a larger number of isolated components and increases the ramification of the force network. This finding is consistent with the results obtained for isotropically compressed systems, where the peak of  $\beta_0$  curves was shown to increase with compression [18].

While additional information could be extracted from other measures, the number of components as a function of force threshold already shows that the properties of force networks are depth-dependent. Therefore, in the rest of this paper, we focus only on the bottom slices, so that various comparisons are not obscured by heterogeneity.

#### B. Structural differences in force networks for the systems exposed to different tap intensities

In this section, we attempt to distinguish properties of the force networks corresponding to different  $\Gamma$ 's that lead to the same steady state packing fractions for disks. The mean stress tensor in the bottom slice of the system differs slightly for the steady states at different  $\Gamma$  [32]. In particular, the mean pressure  $p = (\sigma_1 + \sigma_2)/2$  (where  $\sigma_1$  and  $\sigma_2$  are the principal components of the stress tensor) is  $(24.5 \pm 0.4)mg/d$  for low taps and  $(25.7 \pm 0.4)mg/d$  for high taps. (Note that in 2D the stress has units of force per unit length.) The deviatoric stress  $q = (\sigma_1 - \sigma_2)/2$ , normalized by the mean pressure  $p$ , is  $q/p = 0.14 \pm 0.01$  for low tap intensity and  $q/p = 0.08 \pm 0.01$  for high tapping. Hence, the states are distinguishable through macroscopic properties. However, our goal is to distinguish the force networks after normalizing by the average force.

Figure 5 shows the PDFs for the systems composed of disks, for both normal and tangential forces. Note that they do not show any significant differences. This is consistent with the results of similar simulations reported in Ref. [17]. The PDFs are unaffected by the value of  $\Gamma$  at least as long as they lead to the same  $\phi$ .

We note that for low tap intensity, the particles barely move, while for high tapping intensity the particles completely rearrange between the consecutive taps. Recalling further that

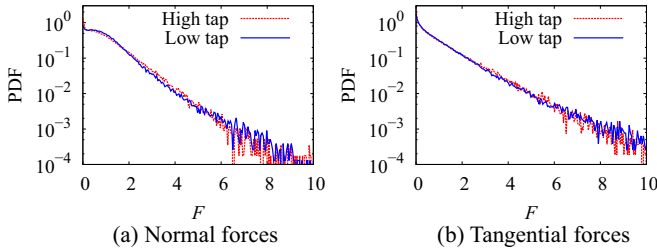


FIG. 5. PDFs (disks, bottom slice).

we are considering monodisperse disks, one could expect that a certain degree of ordering or crystallization occurs in the system tapped at low intensity. This may lead to long-range correlation in the contact forces that may complicate the comparison with disordered structures such as the ones observed in polydisperse systems or in packings of noncircular grains.

Figure 6 shows the contact–contact pair correlation function,  $g(r)$ . The  $g(r)$  is calculated as usual (see, e.g., Ref. [44]) but using the position of contact points instead of the centers of the particles. To avoid boundary effects on  $g(r)$  due to the empty spaces outside the containing box, we only average over particles in a central area of the slice, away from the walls. Figure 6 shows that there are indeed long-range correlations in the disk packings, and these correlations are, as expected, much stronger for low intensity taps. The first three peaks in the  $g(r)$  correspond to the three typical distances between the six contacts of a circular grain in a triangular lattice arrangement.

To assess the impact of spatial ordering on the properties of force distribution, we consider the correlation of the contact forces by calculating the force-force correlation function,  $f(r)$ , defined as

$$f(r) = \frac{1}{\rho_c} \sum_{i,j>i} \delta(r - r_{i,j}) f_i f_j,$$

where  $\rho_c$  is the density of contact points,  $\delta$  is the Dirac delta distribution,  $r_{i,j}$  is the contact-to-contact distance, and  $f_i$  is the force experienced at the contact  $i$ , normalized by the average force  $\langle f \rangle$ . The sum runs over all pairs of contacts.

Figure 7 shows  $f(r)$  for disks for the low tap intensity. The overall shape of  $f(r)$  agrees with  $g(r)$  since the main

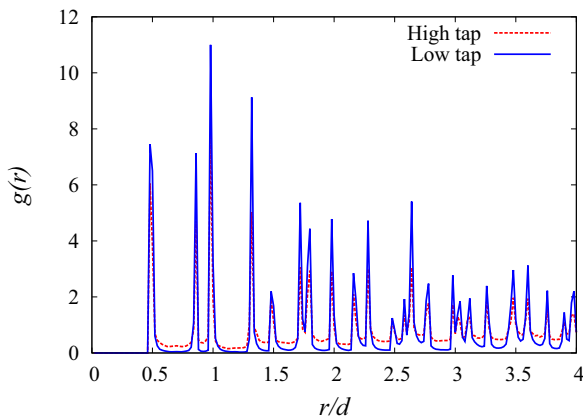


FIG. 6. Pair correlation function,  $g(r)$  (disks, bottom slice).

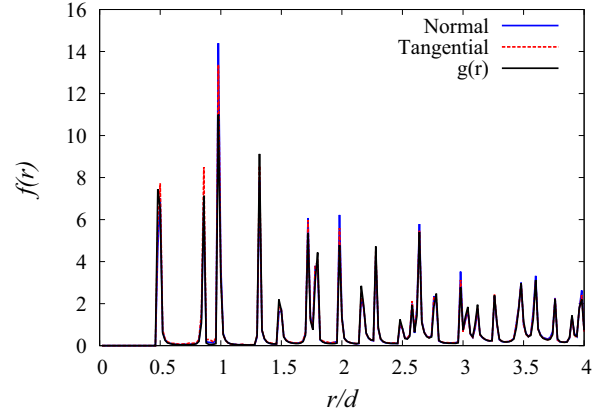


FIG. 7. Force correlation function,  $f(r)$  (disks, low tapping). The  $g(r)$  is also shown.

correlation comes from the positional order of the contact points. To highlight the contribution due to the strength of the forces, we consider the ratio  $f(r)/g(r)$ , shown in Fig. 8, for both low and high tap intensities. There is much less structure in the force correlation if the positional correlations are eliminated. Besides, there is no substantial difference between the considered low and high tap intensities. Therefore, positional ordering (crystallization) does not reflect itself in an ordering of the force strengths. The only strong peak in  $f(r)/g(r)$ , at  $r/d = 1$ , indicates that two contacts on a single grain that are acting from opposite sides have a large probability of having strong forces. However, this is only a short-range correlation.

Now we compare structural differences of the force networks corresponding to the systems prepared with different tap intensities. As can be seen in Fig. 9, counting the number of components,  $\beta_0$ , and the number of loops,  $\beta_1$ , present in different networks does not reveal any remarkable difference between these regimes. For brevity, we show only the count of nontrivial loops. However, including the trivial, three-particle loops does not distinguish the systems either (figures not shown for brevity).

In summary, strong correlations of particle positions do not necessarily lead to correlations of the forces between the particles. Furthermore, none of the measures considered (PDFs, force correlation functions,  $\beta_0$ ,  $\beta_1$ ) uncover any significant differences between the networks of steady states characterized by the same  $\phi$  but obtained using different tap intensities. It is worth mentioning that previous simulations

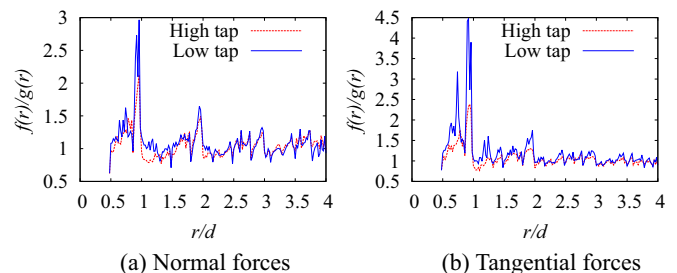


FIG. 8.  $f(r)/g(r)$  (disks, bottom slice).

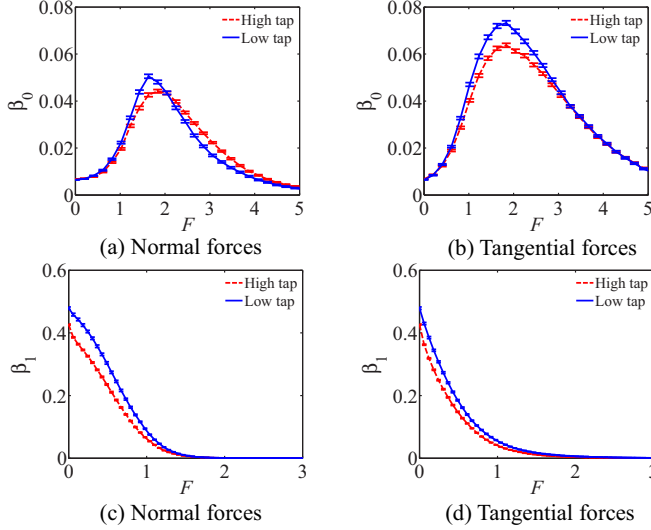


FIG. 9. (a) and (b)  $\beta_0$ , and (c) and (d)  $\beta_1$  (disks, bottom slice). Error bars are as in Fig. 4.

and experiments [14,45] have found some differences in the contact networks (i.e., the force network with  $F = 0$ ) by considering the different loop sizes (trivial loops seem to be numerous in the high tap states). We notice that in the current work walls present virtually zero friction whereas previous studies were done with frictional walls. Also, we focus on the properties of a horizontal layer of grains in contrast with the analysis of the entire column done previously. Based on our results, we conclude that a more careful analysis of the structure of the force networks is needed for the purpose of finding the differences between these equal- $\phi$  states. Such analysis is presented in Ref. [30].

### C. Structural differences of the force networks for disks and pentagons

Understanding differences between force networks in the systems composed of particles of different shapes is important to assess to what extent simple models based on circular or spherical grains capture the properties of realistic systems. In the following, we will discuss the differences between the systems built from disks and pentagons; for brevity, we consider only the low tapping regime.

In addition to the discussion of force PDFs and structural properties of force networks, we have also considered the stress tensor for the considered systems, to find that the disks lead to a higher mean pressure,  $p = (24.5 \pm 0.4)mg/d$ , compared to the pentagons,  $p = (22.8 \pm 0.4)mg/d$ . Also, we note that the earlier studies of the contact forces for elongated particles [25,26], pentagons [27], and irregular polyhedral particles [22] revealed that the orientational distribution of the contact forces was more anisotropic than for circular or spherical grains. For the considered systems, we find that the mean deviatoric stress normalized by the mean pressure is  $q/p = 0.14 \pm 0.01$  for disks and  $q/p = 0.24 \pm 0.01$  for pentagons. This indicates a more anisotropic stress for pentagons, consistent with results on isotropically compressed systems [27]. Again, these results reveal that there are some differences between the systems,

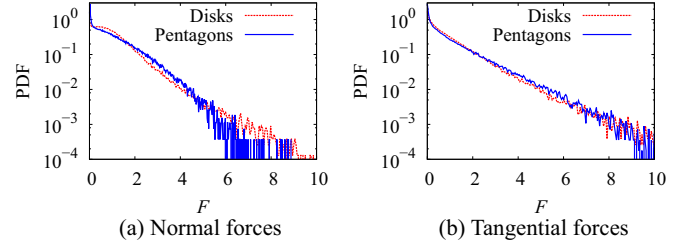


FIG. 10. PDFs (bottom slice, low tapping).

yet provide no information about the connectivity of the force network.

We start by considering the PDFs for disks and pentagons, shown in Fig. 10. Note that the distributions of the tangential forces are indistinguishable. On the other hand, there is a faster decay at large forces for pentagons. Disks contain a larger number of contacts than pentagons at normal forces around  $F = 1$ . This is in agreement with the results for similar systems exposed to isotropic compression [27]. Let us now consider the behavior of the PDF of the normal forces for  $F > 1$ . For pentagons, the decay is consistent with Gaussian behavior while for the disks the decay is exponential. A Gaussian decay has been shown to be connected with the presence of arches in the structure [17]. This is consistent with the intuition that pentagons are more prone to form arches than circular grains. However, the differences in PDF are rather subtle. Moreover, disk packings can also show Gaussian-like PDF depending on the preparation protocol [46]; hence a Gaussian PDF is not a signature of noncircular particles. In the rest of this section, we show that more significant differences between the systems can be identified by considering the global geometry of the force networks.

Figure 11 shows  $\beta_0$  as a function of the force threshold,  $F$ , for disks and pentagons. For normal forces,  $\beta_0$  is rather similar for both types of particles. In the range  $2 < F < 4$ , the number of components is slightly larger for pentagons. This shows a correlation with the PDFs given in Fig. 10(a) where pentagons show a larger number of contacts in the same range. Note that these features do not need to be correlated since  $\beta_0$  depends not only on the number of contacts at a particular force level, but also on how these contacts (edges) connect with the edges in the portion of the force network that corresponds to larger  $F$ . In general,  $\beta_0$  is larger for tangential than for normal forces [compare Figs. 11(a) and 11(b)]. This indicates that the edges of the normal force network corresponding to

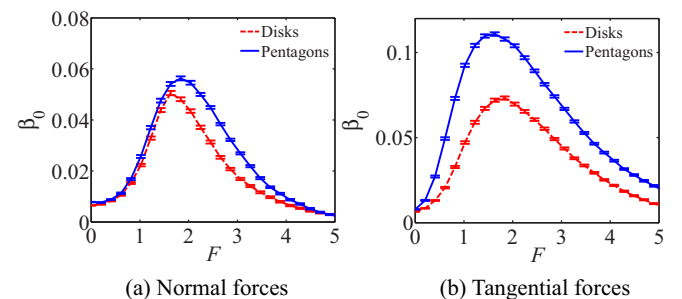


FIG. 11.  $\beta_0$  (bottom slice, low tapping).

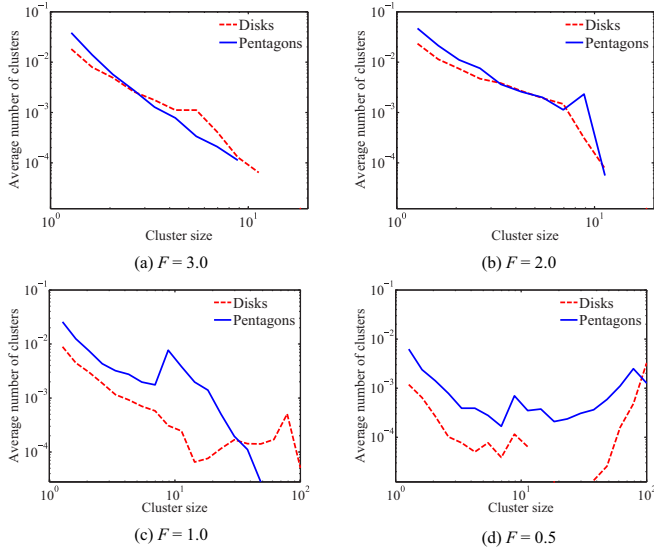


FIG. 12. Average number of components (clusters) per particle for disks and pentagons in the tangential force network as a function of cluster size for various force thresholds,  $F$ . (a)  $F = 3$ , (b)  $F = 2.0$ , (c)  $F = 1.0$ , and (d)  $F = 0.5$  (bottom slice, low tapping).

the contacts characterized by strong forces tend to be more connected and form fewer clusters than for the tangential force network. Hence, for the normal force network, the edges with lower force values are more likely to be connected to the components present in the force network for larger force threshold. This effect is observed in Fig. 1, where force chains (corresponding in the loose sense to the components at high  $F$ ) are larger for the normal than for the tangential forces.

Restricting our attention to the tangential forces, Fig. 11(b) indicates that  $\beta_0$  is significantly larger for the pentagons than for disks. This means that, as we reduce the threshold  $F$  from a large initial value, the new contacts that come into play in the tangential force network tend to be more disconnected from the previous high force contacts for pentagons, in comparison to disks.

In order to highlight further differences between the systems composed of pentagons and disks, we focus on the tangential force networks, since these show clearer contrasts. We start by investigating the size of the components at different force thresholds. Figure 12 shows the average number of components, per particle, as a function of component size for four different values of  $F$ . The shape of the curves changes significantly around  $F = 2$ . Around this point, the total number of components,  $\beta_0$ , for both systems reaches its maximum [see Fig. 11(b)]. For  $F \geq 2$ , the pentagon-base system tends to have a larger number of small clusters, consisting of less than approximately 10 particles (see Fig. 12), while the disk-based system contains a larger number of larger clusters. This suggests that for the disks, the contacts with high forces tend to aggregate (in larger clusters). These high force contacts are more scattered for the pentagons.

As the value of  $F$  is lowered below 2, the components start merging together. Naturally, the number of small clusters decreases as they merge and create larger ones. However, there is a substantial difference between disks and pentagons.

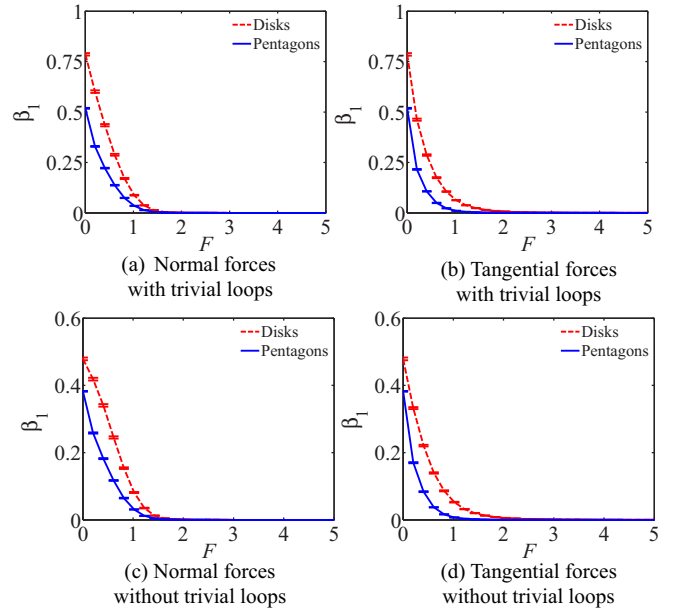


FIG. 13.  $\beta_1$  as a function of the force threshold  $F$  for disks and pentagons. The trivial loops are included in (a) and (b) and omitted in (c) and (d) (bottom slice, low tapping).

For  $F = 1$ , pentagon packings do not contain any clusters composed of more than  $\approx 50$  particles, while for the disks a cluster of size comparable to the number of particles in the slice can be formed, *viz.*, Fig. 12(c). Again, the number of small clusters is larger for pentagons. Finally, for  $F = 0.5$  the disks do not form any clusters of intermediate sizes. This is not the case for pentagons, which exhibit clusters of all possible sizes.

We conclude that disks have a more heterogeneous structure of the tangential force network than pentagons. In disk packings, contacts with large tangential forces tend to appear spatially clustered, and these clusters merge together for relatively large values of  $F$ . On the other hand, pentagons form a large number of small spatially scattered clusters of high tangential forces. These scattered clusters grow and merge without spanning the system until  $F$  reaches a very low value.

Let us now consider the number of loops in the force networks described by the first Betti number,  $\beta_1$ . Figures 13(a)–13(b) show the number of all loops as a function of  $F$ . Figures 13(c)–13(d) present similar data, but here the trivial loops formed by three particles in contact are not included. In all the cases  $\beta_1$  decays rather fast, and there are only a few loops for  $F > 2$ . This is as expected since for  $F > 2$  the clusters tend to be too small to contain a loop.

Disk packings contain a larger number of loops than pentagons for both normal and tangential force. The number of trivial loops is significantly larger for the disks because of their tendency to form crystalline regions. The larger number of loops observed in the tangential force network for disks even at  $F > 2$  is connected with the fact that these networks typically contain larger clusters that support a larger number of loops. The number and sizes of the clusters in the normal force network are similar for both disks and pentagons, and thus, the reason for a larger number of loops in the normal force network

for the disks is less clear. We note that the differences in the loop structure of the force networks for disks and pentagons are consistent with the intuition that pentagons tend to form long arches that create large loops (loops of many edges) in the force network. Due to the large size of the loops their number tends to be smaller. Finally, for  $F = 0$ , the number of loops in both normal and tangential force networks equals the number of loops in the contact network. We note that the number of loops present in the contact network for disks is consistent with an earlier study [14].

#### IV. CONCLUSIONS

We have studied the force network of static packings under gravity by averaging properties over a collection of realizations obtained by tapping. We focus on exploring significant differences of the force network when particles of different shapes (disks versus pentagons) are considered, but we have also explored difference between states obtained by using different tap intensities.

We have shown that the topology of the force network changes with depth. To some degree, this can be inferred from the PDFs of the contact forces. However, the detailed analysis of components sheds further light on this phenomenon. The differences are not only in the average pressure, but also in the structural properties of the network. Therefore, in this paper we analyze slices of the column to avoid artifacts due to the vertical heterogeneity of the network.

We also compare realizations obtained using very different tap intensities that nevertheless have the same average packing fractions. These steady states have been shown previously to present distinguishable mean stress [32] and distinguishable number of trivial loops in the contact network [14,45], albeit by using a different protocol and frictional walls. For the present system, our results for the PDFs of the contact forces, force-force correlations, number of clusters and loops, cannot set apart in any significant way these equal-density states.

Comparison of packings of particles of different shape has revealed that pentagons tend to exhibit a more homogeneous tangential force network than disks. For the pentagon packings, the edges in the force network, corresponding to the strong contacts, are more spatially scattered and less connected to each other. In contrast, for the disks, the edges with smaller force value tend to be directly connected to the edges with larger values. Hence, the part of the tangential force network exceeding any given force threshold contains a larger number of small components for pentagons than for disks. In particular, pentagon packings contain many scattered “hot spots” in the tangential force network. On the other hand, for the disk packings, the part of the tangential force network exceeding larger values of  $F$  consists of smaller number of larger components.

These clear differences in the tangential force network do not have a matching effect in the normal force network. Differences in the normal force network between disks and pentagons are hard to detect. Interestingly, shorter force chains, consistent with smaller clusters, have been observed for pentagons in comparison with disks in experiments involving shearing (see Ref. [29], Fig. 26). Notice however that in this experimental study only normal forces are considered.

One significant difference between disk and pentagon packings is the number of loops. The existence of a larger number of trivial loops (formed by three grains in mutual contact) in the case of disks is not particularly surprising. However, pentagons also display a smaller number of nontrivial loops. This suggests that pentagons tend to form rather large loops, particularly when tangential force networks are considered.

One obvious question is what the presented results say about macroscale response of the various systems considered. A general answer here is that differences on mesoscale clearly influence macroscale behavior; we provided few standard examples of this connection in the introduction. Our results suggest that differences on mesoscale between the considered systems are present but are not always obvious. For example, the listed differences between disks and pentagons are, based on the results presented in the current paper, mostly limited to the tangential forces, and therefore one would expect that friction (that to some degree determines the strength on the tangential forces relative to the normal ones) may play an important role. The connectivity of these tangential force networks is significantly different between disks and pentagons, suggesting that any material response that is influenced by force networks’ connectivity (such as acoustic propagation or elastic response) will be different as well. Understanding how different this response may be will require considering and correlating force networks and macroscale properties of the systems considered in the setup of relevance. In the present work we focus predominantly on identifying these differences.

While the approach used in the present paper has uncovered a number of properties of force networks, it should be pointed out that counting the number of (connected) components and loops as a function of force threshold does not contain the whole picture: knowing Betti numbers does not tell us how the connections of the features (components, loops) evolve as force threshold is varied. For this purpose, we need to consider additional measures, based on persistent homology. As we will see in Ref. [30], this approach uncovers significant additional features of force networks and in particular allows us to quantify the differences between the systems exposed to different tap intensities that lead to the same packing fraction. In addition, persistent homology will allow us to describe and even measure the differences in the force networks from one realization or tap to the next.

In the present work, we have focused on the influence of gravitational compaction, tapping intensities leading to similar packing fraction and particle shape on the properties of the force networks. However, we have not studied the effects of boundary conditions or system size and have not systematically explored the general influence of the tapping intensity on the properties of force networks. These directions of research will guide our further investigations.

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