Non-Lyapunov Control of a Balancing Robot

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Abstract: Applying the necessary and sufficient condition derived in [3] to a balancing unicycle robot, a closed-loop control strategy is presented to stabilize asymptotically the vehicle.

Unlike the classical linearization or even local and specialized strategies, the necessary and sufficient condition applied, render the design straightforward along with simple algebraic conditions to derive the controller.

Keywords: *Balancing robot, Nonlinear system, Asymptotic stability*
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1 Introduction

Asymptotic stability is always a mater of research interest among the control community (see for instance [1]).

As mentioned in [2], dynamical systems modelled as a set of ordinary differential equations (ODE), possess an inherent trajectories knowledge lack.

However, the simple Lyapunov's theorem provides a sufficient condition for stability leaving the user the existence decision by other methods. For this reason a necessary but also sufficient contidion was derived in [3] with simple algebraic conditions which, moreover, reduces systematically the order of the system under study.

In this paper the necessary and sufficient algebraic condition in [3] is applied to stablize vertically a unicycle balaning robot (see for instance [4] for a detialed reading in balancing robots).

This paper is organized as follows: In Section 2 the mathematical model of a unicycle balancing robot using Lagrangian mechanics is presented, Section 3 presents the closed-loop controller derived using the stability condition, Section 4 presents Matlab/Simulink simulations, whereas Section 5 presents some conclusions and future work.

2 Nonlinear Modelling

A simplified nonlinear model of a unicycle like robot with the complete mass carried above (the wheelbase weight is included in that mass above), can be depicted in Figure 1.

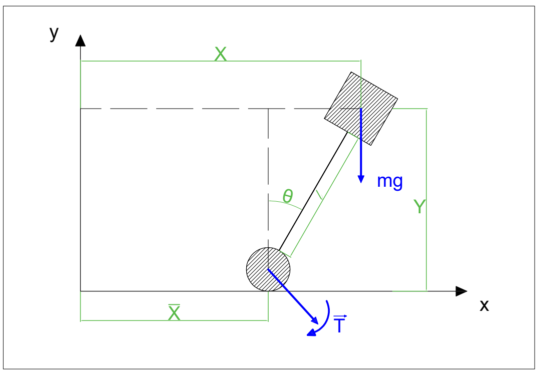

Figure 1: Unicycle-like balancing robot

Lagrangian mechanics leads the following nonlinear control system:

$$L = \frac{1}{2} \cdot M \cdot \left(\dot{X}(t)^2 + \dot{Y}(t)^2 \right) - M \cdot g \cdot (l - Y) \tag{1}$$

where g is the gravity constant, M the complete mass of the robot and the pair $\{X, Y\}$ the position of the center mass of the robot in cartesian coordinates (Figure 1).

Using polar coordinates as the generalized set of coordinates:

$$\begin{cases} X(t) = \bar{X}(t) + l \cdot sin(\theta(t)) \\ Y(t) = l \cdot cos(\theta(t)) \end{cases} \tag{2}$$

where l is the length of the bar rod of the robot. The Lagrangian takes the form:

$$L = \frac{1}{2} \cdot M \cdot \left(\dot{\bar{X}}(t)^2 + 2 \cdot l \cdot cos(\theta) \cdot \dot{\theta} \cdot \dot{\bar{X}} + l^2 \cdot \dot{\theta}^2 \right) - M \cdot g \cdot l \cdot (1 - cos(\theta))$$

Then, the equations of motion (control equations) with a motor torque applied at the wheel-base lead (from (1) and (2)):

$$F \cdot \begin{bmatrix} \ddot{\bar{X}}(t) \\ \ddot{\theta}(t) \end{bmatrix} = G + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \cdot u(t)$$

where $\tau(t)$ is the torque applied by the electric motor at the wheel. Defining the torque $u(t) = \tau(t)$ as the input, along with:

$$F = \begin{bmatrix} 1 & l \cdot cos(\theta) \\ l \cdot cos(\theta) & l^2 \end{bmatrix}, G = \begin{bmatrix} l \cdot sin(\theta) \cdot \dot{\theta}^2 \\ g \cdot l \cdot sin(\theta) \end{bmatrix}$$

Finally:

$$\begin{bmatrix} \ddot{\bar{X}}(t) \\ \ddot{\theta}(t) \end{bmatrix} = F^{-1} \cdot G + F^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \cdot u(t)$$

That is:

$$\begin{bmatrix} \ddot{\bar{X}}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} \frac{l \cdot \dot{\theta}^2 - g \cdot cos(\theta)}{sin(\theta)} \\ \frac{-l \cdot cos(\theta) \cdot \dot{\theta}^2 + g}{l \cdot sin(\theta)} \end{bmatrix} + \begin{bmatrix} -\frac{l \cdot cos(\theta)}{M \cdot l^2 \cdot sin(\theta)^2} \\ \frac{1}{M \cdot l^2 \cdot sin(\theta)^2} \end{bmatrix} \cdot u(t) \tag{3}$$

3 Asymptotic Stability Design

The main tool in this section is the theorem in [3]:

$$\begin{bmatrix} \dot{Z}_1(t) \\ \dot{Z}_2(t) \\ \dot{Z}_3(t) \\ \dot{Z}_4(t) \end{bmatrix} = \begin{bmatrix} Z_2 \\ \frac{l \cdot Z_4^2 - g \cdot cos(Z_3)}{sin(Z_3)} \\ Z_4 \\ \frac{-l \cdot cos(Z_3) \cdot Z_4^2 + g}{l \cdot sin(Z_3)} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{l \cdot cos(Z_3)}{M \cdot l^2 \cdot sin(Z_3)^2} \\ 0 \\ \frac{1}{M \cdot l^2 \cdot sin(Z_3)^2} \end{bmatrix} \cdot u(t)$$

where (3) was transformed to a first order system with $Z(t) = [\bar{X}, \dot{\bar{X}}, \theta, \dot{\theta}]'$ with $'$ the transpose. Then, the following condition must be satisfied in order to derive an asymptotic stabilizing controller (see [3]]):

$$f(V_1) = \alpha \cdot f(V_2), \quad \alpha \neq 0, k \quad \{V_1, V_2\} \in \delta_0$$

where δ_0 is a neighbour of the origin. This condition is equivalent to look for constant direction regions:

$$\Omega = \left\{ X : \frac{f(x)}{\|f(x)\|} = constant \right\}$$

Finally:

$$\begin{aligned}
\frac{Z_2}{\|f(Z)\|} &= \rho_1 = constant\\
\frac{f_1(Z)}{\|f(Z)\|} &= \rho_2 = constant\\
\frac{Z_4}{\|f(Z)\|} &= \rho_3 = constant\\
\frac{f_2(Z)}{\|f(Z)\|} &= \rho_4 = constant
\end{aligned}$$

where $f_1 = \frac{l\cdot Z_4^2 - g\cdot cos(Z_2)}{sin(Z_2)} - \frac{l\cdot cos(Z_2)}{M\cdot l^2\cdot sin(Z_2)^2}\cdot u$ and $f_2 = \frac{-l\cdot cos(Z_2)\cdot Z_4^2 + g}{l\cdot sin(Z_2)} + \frac{1}{M\cdot l^2\cdot sin(Z_2)^2}\cdot u$.
The conditions arise:

$$\begin{aligned}
\frac{Z_2}{f_1(Z)} = \beta_1 = constant \Leftrightarrow &\begin{cases}\dot{Z}_1 = Z_2\\ \dot{Z}_2 = \frac{1}{\beta_1}\cdot Z_2 \Rightarrow \beta_1 < 0\\ \dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}\\
\frac{Z_2}{Z_4} = \beta_2 = constant \Leftrightarrow &\begin{cases}\dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}\\
\frac{Z_2}{f_2(Z)} = \beta_3 = constant \Leftrightarrow &\begin{cases}\dot{Z}_2 = f_1(Z_3, Z_4, u)\\ \dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}\\
\frac{f_1}{Z_4} = \beta_4 = constant \Leftrightarrow &\begin{cases}\dot{Z}_1 = Z_2\\ \dot{Z}_2 = \beta_4\cdot Z_4\\ \dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}\\
\frac{f_1}{f_2} = \beta_5 = constant \Leftrightarrow &\begin{cases}\dot{Z}_1 = Z_2\\ \dot{Z}_2 = \beta_5\cdot f_2(Z_3, Z_4, u)\\ \dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}\\
\frac{Z_4}{f_2} = \beta_6 = constant \Leftrightarrow &\begin{cases}\dot{Z}_1 = Z_2\\ \dot{Z}_2 = f_1(Z_3, Z_4, u)\\ \dot{Z}_3 = Z_4\\ \dot{Z}_4 = \frac{1}{\beta_6}\cdot Z_4 \Rightarrow \beta_6 < 0\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}
\end{aligned} \tag{4}$$

The conditions in (4) can be summarized as follows:

$$\begin{cases}\dot{Z}_3 = Z_4\\ \dot{Z}_4 = f_2(Z_3, Z_4, u)\end{cases} \quad \Leftrightarrow \text{Asymptotically Stable}$$

Along with:

$$\begin{cases} \frac{Z_2}{f_1(Z_3,Z_4,u)} < 0 \\ \frac{Z_4}{f_2(Z_3,Z_4,u)} < 0 \end{cases}$$

With these conditions, it is not difficult to derive a control law to stabilize $\{Z_3, Z_4\}$ independently of the pair: $\{Z_1, Z_2\}$, however, the longitudinal dynamics can be considered out of the stability scope, so it is possible to define:

$$u = M \cdot l^2 \cdot sin(Z_3) \cdot \left(\frac{-l \cdot cos(Z_3) \cdot Z_4^2 + g}{l \cdot sin(Z_3)} + (a_1 \cdot Z_3 + a_2 \cdot Z_4) \right), \quad \{a_1, a_2\} < 0 \tag{5}$$

4 Matlab Simulations

Incorporating the control law (5) into the model (3), the simulations in Matlab/Simulink are depicted in Figure 2 with $a_1 = -10, a_2 = -10$.

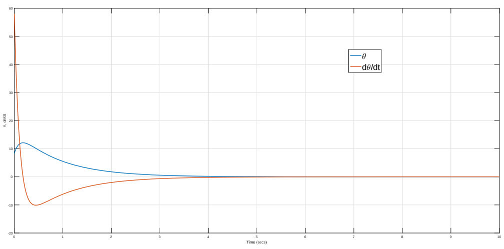

Figure 2: Asymptotic stability for unicycle-like balancing robot: $l = 0.15, M = 1000$

It is interesting to note that the initial condition: $Z(0) = \{0.150 \quad 3.5 \quad 8.3 \quad 58\}$ is far away from the origin, so the nonlnear technique derived in [3] since to work no only locally.

5 Conclusions

A nonlinear control law was derived to control a balancing robot vertically . The local nature of the control law seems to verify a more global condition in the view of the wide range of initial conditions stabilized.

While the balancing robot was used as a benchmark to test this non-Lyapunov control technique, a future work to extend the local condition a global stabilizing technique is envisioned.

Acknowledgments

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